Recent Study of Semigroups (準群の代数的理論)

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Recent Study of Semigroups*)

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This note will introduce the results of the study of semigroups which were obtained by a group containing the author in recent years. Some of them are already published, but some are not published yet. Because of space the detailed proof will be omitted here. One of the topics is concerned with the study of the structure of S-indecomposable semigroups; another is a new concept "general product" which is related with the problems of extensions and compositions of semigroups.

1. Decomposition

Let ρ be a congruence on a semigroup S. If S/ρ satisfies a system of identities or, more generally, a system $\mathcal{J}$ of implication, ρ is called a $\mathcal{J}$-congruence on S. For any system $\mathcal{J}$ of implications and for any semigroup S there is a smallest $\mathcal{J}$-congruence ρ_o on S (see [2]). The partition of S which is due to a $\mathcal{J}$-congruence ρ is called a $\mathcal{J}$-decomposition of S and the partition due to ρ_o is called the greatest $\mathcal{J}$-decomposition of S. A semigroup which is isomorphic with S/ρ is called a $\mathcal{J}$-homomorphic image of S and a semigroup isomorphic with S/ρ_o is called the greatest $\mathcal{J}$-homomorphic image of S. If the cardinality $|S/\rho_o|$ of S/ρ_o is 1, then S is said to be $\mathcal{J}$-indecomposable; otherwise $\mathcal{J}$-decomposable.

The simplest examples of a system $\mathcal{J}$ of identities are:

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*) This note is to supplement the materials spoken by the author at the Symposium, Kyoto University, in June 1967.

$$C = \{xy = yx\},\ \mathcal{J} = \{x^2 = x\},\ S = \{xy = yx,\ x^2 = x\}.$$

Theorem 1. In the greatest S-decomposition of a semigroup, each congruence class is S-indecomposable.

This theorem was originally obtained by the author [17]. To prove the theorem the following fact was used. Let S be a semigroup and let

(1) $S = \bigcup_{\lambda\in\Gamma} S_\lambda$

be an S-decomposition of S which need not be greatest. Let ρ denote the congruence on S due to the decomposition (1). Let σ be an S-congruence on S_λ. Then there exists an S-congruence τ on S such that the restriction $\tau|S_\lambda$ coincides with σ. Consequently if ρ is smallest, then S_λ is S-indecomposable.

In 1964 M. Petrich gave another proof of the theorem by using the concept of prime ideals and faces [9]. At that time the author introduced "content" as follows:

Let S be a semigroup and $a_1,a_2,\cdots,a_n$ be a finite number of elements of S. Consider the set C of all elements a of S which is the product of $a_1,\cdots,a_n$ admitting repeated use:

$$a = a_{i_1}^{k_1} a_{i_2}^{k_2} \cdots a_{i_m}^{k_m}$$

where the set $\{i_1,i_2,\cdots,i_m\}$ coincides with $\{1,2,\cdots,n\}$. C is a subsemigroup of S. C is called a content of $a_1,\cdots,a_n$ in S, and denoted by $C = C\langle a_1,\cdots,a_n\rangle$. It is interesting that every content is S-indecomposable. By using this result, the author gave the third proof of Theorem 1 in [21].

If S is commutative, S is S-indecomposable if and only if it is archimedean, that is, for every $a, b \in S$ there are $x, y \in S$ and positive integers m, n such that

$$a^m = bx, \; b^n = ay.$$

A commutative archimedean semigroup contains at most one idempotent element (see [1]).

2. Introduction to Commutative Archimedean Semigroups

We can classify all commutative archimedean semigroups into the four types:

Type 1. Commutative nil-semigroups (i.e. having unique idempotent which is a zero).

Type 2. Commutative unipotent semigroups without zero.

Type 3. $\mathfrak{N}$-semigroups (i.e. commutative archimedean cancellative semigroups without idempotent).

Type 4. Commutative archimedean semigroups without idempotent.

The class of Type 4 contains the class of Type 3. The terminology "$\mathfrak{N}$-semigroup" is due to Petrich [8].

The study of semigroups of Type 1 is reduced to that of commutative semigroups satisfying $x^2 = 0$ for all x. If S is a commutative semigroup of Type 1, then for every $a \in S$ there is a positive integer m depending on a such that $a^m = 0$, and S is the union of ascending chain of semigroups

$$S_i: S = \bigcup_{i=1}^{\infty} S_i, \; S_1 \subseteq S_2 \subseteq \cdots$$

where each S_i is an ideal of S and it satisfies $x^2 = 0$ for all $x \in S_i$, and the Rees factor semigroup S_{i+1}/S_i also satisfies $x^2 = 0$ for all x.

The structure of Type 2 is well-known as the ideal extension of group by a semigroup of Type 1. (cf. [14], [15], [19].)

As far as Type 3 is concerned,

Theorem 2. [18] Let G be an abelian group, N the set of all non-negative integers. Let $I(\alpha,\beta)$ be a function: $G \times G \to N$ which satisfies the following four conditions:

(2.1) $I(\alpha,\beta) = I(\beta,\alpha)$.

(2.2) $I(\alpha,\beta) + I(\alpha\beta,\gamma) = I(\alpha,\beta\gamma) + I(\beta,\gamma)$.

(2.3) For any $\xi \in G$ there is a positive integer m such that $I(\xi^m,\xi) > 0$.

(2.4) $I(\varepsilon,\varepsilon) = 1$.

Let $S = \{(n,\alpha);\ n \in N,\ \alpha \in G\}$. We define a binary operation on S as follows:

$$(n,\alpha)(m,\beta) = (n + m + I(\alpha,\beta),\ \alpha\beta).$$

Then S is an $\mathcal{N}$-semigroup. Every $\mathcal{N}$-semigroup is obtained in this manner. G is called the structure group of S.

When the author obtained Theorem 2 in 1957 he left the two problems unsolved. One problem is: Under what condition on G, G', I, I', are (G,I) and (G',I') isomorphic? Another problem is to find all the functions I on a given G. M. Sasaki [10] and J. Higgins [5] gave answers to the first problem, and Sasaki and the author [1] partially solved the second problem. Sasaki will discuss precisely the $\mathcal{N}$-semigroups in [12].

Type 4, the case where cancellation is not assumed, has been studied by the author [24].

At any rate, every semigroup is the set union of a special type of subsemigroups, namely S-indecomposable semigroups. A question is naturally raised: Are there any other types except "semilattice" for which a semigroup is the set union of a special type of subsemigroups? In fact "semilattice" is a unique system of identities with such a property in the following sense.

Let $\mathcal{J}$ be a system of identities $f_\lambda(x_1,\cdots,x_n) = g_\lambda(x_1,\cdots,x_n)$. Any semigroup S has a greatest $\mathcal{J}$-decomposition. Suppose that each congruence cl. of S - if it is a subsemigroup - is $\mathcal{J}$-indecomposable. Then $\mathcal{J}$ is called an attainable system of identities on all semigroups. Of course both $\{x = x\}$ and $\{x = y\}$ are attainable. They are called trivial.

The author proved the following theorem [22]:

Theorem 3. The semilattice $\{x = x^2, xy = yx\}$ is a unique non-trivial attainable system of identities on all semigroups.

This means that if $\mathcal{J}$ is a non-trivial attainable system of identities on all semigroups, $\mathcal{J}$ is equivalent to $\{x = x^2, xy = yx\}$. Thus the significance of greatest S-decomposition is again understood in the study of the structure of semigroups in the following sense: The study of semigroups is reduced to the study of the structure of S-indecomposable semigroups and the compositions of S-indecomposable semigroups.

3. Commutative Archimedean Semigroups of Type 4

Definition. Let L be a lower semilattice (i.e. partially ordered set in which every two elements have a greatest lower bound). L is called a discrete tree if it satisfies the following conditions:

(3.1) $x < z$, $y < z \Rightarrow x \leqq y$ or $x \geqq y$.

(3.2) For any b, c $\in$ L such that $b \leq c$, the set $\{x;\ b \leqq x \leqq c\}$ is finite.

Let P be a set and $N = \{0,1,2,\cdots\}$. Consider a mapping $P \times P \to N \times N$

$$(p,q) \to (h_p(p,q),\ h_q(p,q))$$

satisfying

(4.1) $h_p(p,q) \geqq 0,\ h_p(p,q) = 0 \iff p = q.$

(4.2) $h_p(p,q) = h_p(q,p).$

(4.3) For every three elements $p, q, r \in P$, one of (4.3.1), (4.3.2) and (4.3.3) holds:

$$(4.3.1)\quad \begin{cases} h_p(r,p) + h_q(p,q) = h_p(p,q) + h_q(q,r), \\ h_q(q,r) \geqq h_q(q,p),\ h_r(r,p) = h_r(q,r). \end{cases}$$

$$(4.3.2)\quad \begin{cases} h_q(p,q) + h_r(q,r) = h_q(q,r) + h_r(r,p), \\ h_r(r,p) \geqq h_r(r,q),\ h_p(p,q) = h_p(r,p). \end{cases}$$

$$(4.3.3)\quad \begin{cases} h_r(q,r) + h_p(r,p) = h_r(r,p) + h_p(p,q), \\ h_p(p,q) \geqq h_p(p,r),\ h_q(q,r) = h_q(p,q). \end{cases}$$

Let $L' = \{(n,p);\ n \in N,\ p \in P\}$. We define relations ζ and ξ as follows:

$$(m,q)\zeta(n,p) \iff m-h_q(p,q) \geqq 0,\ n-h_p(p,q) \leqq m-h_q(p,q).$$

$$(m,q)\xi(n,p) \iff (m,q)\zeta(n,p) \text{ and } (n,p)\zeta(m,q).$$

Then L'/ξ is a discrete tree with respect to a partial order ζ/ξ; it satisfies the ascending chain condition and contains no least element (see [16]).

Each element of the form $(0,p)$, $p \in P$, is a maximal element, which will be called also a prime (element). Identifying p with $(0,p)$, p is regarded as a prime of L.

Let p_α be any prime and p_o be a fixed prime of L. Define the notations:

$$(5)\quad \begin{cases} \pi_{p_o}(p_\alpha) = h_{p_o}(p_o,p_\alpha),\ \tilde{\pi}_{p_o}(p_\alpha) = h_{p_\alpha}(p_o,p_\alpha). \\ \sigma_{p_o}(p_\alpha) = \pi_{p_o}(p_\alpha) - \tilde{\pi}_{p_o}(p_\alpha). \end{cases}$$

Definition. If $\sigma_{p_o}(p_\alpha) \geqq 0$ for all primes p_α, then the prime p_o is called a highest prime (highest maximal element). A discrete tree L is called an ordinary tree if L satisfies the ascending chain condition and has at least one highest maximal element.

Let S be a commutative archimedean semigroup without idempotent. For a fixed element $a \in S$ (we call a a standard element), we define relations ρ_a and τ_a as follows:

$$x \rho_a y \iff a^n x = a^m y \quad \text{for some} \quad n, m > 0.$$

$$x \tau_a y \iff x = a^n y \quad \text{for some} \quad n > 0.$$

Then S/ρ_a is an abelian group and each congruence class modulo ρ_a is an ordinary tree without least element. We define a relation η on S as follows:

$$x \eta y \iff a^n x = a^n y \quad \text{for some} \quad n > 0.$$

Then S/η is an $\mathfrak{N}$-semigroup.

(6.1) Let G be an abelian group with $I(\alpha,\beta)$ on $G \times G$ satisfying (2.1), (2.2), (2.3) and (2.4),

(6.2) $\{S_\lambda;\ \lambda \in G\}$ a system of ordinary trees S_λ of infinite length and $\{\iota_\lambda;\ \lambda \in G\}$ a system of highest primes ι_λ, exactly one ι_λ from S_λ.

(6.3) $P = \bigcup_{\lambda \in G} P_\lambda$ the set of all primes of S, P_λ the set of all primes of P_λ. Suppose that a commutative binary operation $(\cdot)$ is defined on P such that ι_ε is the identity element, and if $\alpha_\lambda \in P_\lambda$ and $\beta_\mu \in P_\mu$, then $\alpha_\lambda \cdot \beta_\mu \in P_{\lambda\mu}$ and the following three conditions are satisfied: For all

α_λ, β_μ, γ_ν, β'_μ, all λ, μ, ν,

(6.3.1) $\sigma(\alpha_\lambda) + \sigma(\beta_\mu) + I(\lambda,\mu) - \sigma(\alpha_\lambda \cdot \beta_\mu) \geqq h_{\alpha_\lambda \cdot \beta_\mu}(\alpha_\lambda \cdot \beta_\mu, \alpha_\lambda \cdot \beta'_\mu) - h_\beta(\beta_\mu, \beta'_\mu)$.

(6.3.2) $\sigma(\alpha_\lambda) + \sigma(\beta_\mu) + \sigma(\gamma_\nu) + I(\lambda,\mu) + I(\lambda\mu,\nu) \geqq \sigma((\alpha_\lambda \cdot \beta_\mu) \cdot \gamma_\nu) + h_{(\alpha_\lambda \cdot \beta_\mu) \cdot \gamma_\nu}((\alpha_\lambda \cdot \beta_\mu) \cdot \gamma_\nu, \alpha_\lambda \cdot (\beta_\mu \cdot \gamma_\nu))$.

(6.3.3) For any $\alpha_\lambda \in P_\lambda$, there is a positive integer m such that

$$\sigma(\alpha_\lambda^{(m)}) + \sigma(\alpha_\lambda) + I(\lambda^m, \lambda) - \sigma(\alpha_\lambda^{(m)} \cdot \alpha_\lambda) > 0$$

where $\alpha_\lambda^{(m)} = \alpha_\lambda^{(m-1)} \cdot \alpha_\lambda$, $\alpha_\lambda^{(2)} = \alpha_\lambda \cdot \alpha_\lambda$.

Then a 5-tuple $((G,I;\ \{S_\lambda\},\ \{\iota_\lambda\};\ P))$ is called a structural system.

Theorem 4. Suppose a structural system $((G,I;\ \{S_\lambda\},\ \{\iota_\lambda\};\ P))$ is given. Let

$$S'_\lambda = \{(n,\alpha);\ n=0,1,\cdots,\ \alpha_\lambda \in P_\lambda\},$$

and

$$S' = \bigcup_{\lambda \in G} S'_\lambda.$$

Define an equivalence relation ξ on S' by

$$(n,\alpha)\,\xi(m,\beta) \iff \begin{cases} \alpha_\lambda \in P_\lambda,\ \beta \in P_\lambda. \\ \text{and } n-h_\alpha(\alpha,\beta) = m-h_\beta(\alpha,\beta) \geqq 0. \end{cases}$$

Let $S = S'/\xi$, and let

$$K(\alpha_\lambda,\beta_\mu) = \sigma(\alpha_\lambda) + \sigma(\beta_\mu) + I(\lambda,\mu) - \sigma(\alpha_\lambda \cdot \beta_\mu).$$

A binary operation is defined on S as follows:

Let $[(n,\alpha_\lambda)]$ denote an element of S i.e. the ξ-equivalence class in

S' which contains (n,α_λ).

(7) $[(n,\alpha_\lambda)][(m,\beta_\mu)] = [(n + m + K(\alpha_\lambda,\beta_\mu),\ \alpha_\lambda\cdot\beta_\mu)]$.

Then S is a commutative archimedean semigroup without idempotent. Every commutative archimedean semigroup without idempotent can be obtained in this manner.

We have obtained a structure theorem, but many questions still remain. One of them is this: Given G, I and a system $\{S_\lambda\}$ of ordinary trees of infinite length with $\{\iota_\lambda\}$ assigned, does there exist a commutative groupoid P? If so, how can we determine all those? This problem is still open in general except some special cases.

Theorem 5. The following condition implies (6.3.1), (6.3.2) and (6.3.3).

(8) $\sigma(\alpha_\lambda) + \sigma(\beta_\mu) + I(\lambda,\mu) \geqq \pi(\alpha_\lambda\cdot\beta_\mu)$ for all α_λ, β_μ, all λ, μ, which is equivalent to (9.1) and (9.2).

(9.1) If $\alpha_\lambda \neq \iota_\lambda$ or $\beta_\mu \neq \iota_\mu$ then $\sigma(\alpha_\lambda) + \sigma(\beta_\mu) + I(\lambda,\mu) - \sigma(\alpha_\lambda\cdot\beta_\mu) \geqq 1$.

(9.2) $\widetilde{\pi}(\alpha_\lambda) = 1$ for all $\alpha_\lambda \neq \iota_\lambda$, all $\lambda \in G$.

Definition. If an ordinary tree L satisfies (9.2) then L is called a sparse tree. If $((G,I;\ \{S_\lambda\},\ \{\iota_\lambda\};\ P))$ satisfies (9.1) and (9.2), then it is called a sparse structural system.

We have the existence theorems with respect to sparse structural systems in two ways.

Theorem 6. Suppose that an abelian group G and a function I is given and that $\{S_\lambda;\ \lambda \in G\}$ is a family of disjoint sparse trees of infinite length. Let P_λ be the set of all primes of S_λ and $\{\iota_\lambda;\ \lambda \in G\}$ be a representative system of highest primes. Then there is a commutative groupoid $P = \bigcup_{\lambda\in G} P_\lambda$ with identity element ι_ε such that $((G,I;\ \{S_\lambda\},\ \{\iota_\lambda\};\ P))$ is a sparse structural system.

Theorem 7. Suppose that an abelian group G is given and that there is given a family of disjoint sets $\{P_\lambda;\ \lambda \in G\}$ in which a system $\{\iota_\lambda;\ \lambda \in G\}$, $\iota_\lambda \in P_\lambda$, is assigned. Let $P = \bigcup_{\lambda \in G} P_\lambda$. Also assume that a groupoid $(\cdot)$ is given on P such that

$$\alpha_\lambda \cdot \beta_\mu = \beta_\mu \cdot \alpha_\lambda \in P_{\lambda\mu} \quad \text{for all} \quad \alpha_\lambda \in P_\lambda,\ \beta_\mu \in P_\mu, \quad \text{all} \quad \lambda, \mu \in G,$$
$$\alpha_\lambda \cdot \iota_\varepsilon = \iota_\varepsilon \cdot \alpha_\lambda = \alpha_\lambda \quad \text{for all} \quad \alpha_\lambda \in P_\lambda, \quad \text{all} \quad \lambda \in G.$$

Then there is a function I and a family $\{S_\lambda;\ \lambda \in G\}$ of sparse trees of infinite length such that P_λ is the set of all primes of S_λ in which ι_λ is highest and $((G,I;\ \{S_\lambda\},\ \{\iota_\lambda\};\ P))$ is a sparse structural system.

Definition. A commutative archimedean semigroup S is called jointed if and only if there is an element p such that for every element x of S, $p^m = p^n x$ for some m, $n > 0$ depending on x. Related to a jointed commutative archimedean semigroup without idempotent, $((G,I;\ \{S_\lambda\},\ \{\iota_\lambda\};\ P))$ in which $G = \{\varepsilon\}$, $I(\varepsilon,\varepsilon) = 1$, is called a jointed structural system and it is denoted by $((S,\iota;\ P))$.

Theorem 8. Let S be an ordinary tree of infinite length and let ι be a highest prime of S. Then there is a commutative groupoid P defined on the set of all primes of S such that $((S,\iota;\ P))$ is a jointed structural system.

As seen in Theorem 4 the conditions seem to be complicated since the class the theorem treats is large. Is it possible to improve the conditions, though it might be necessary to restrict ourselves to a smaller class?

The detailed discussion in this section will appear in [24].

4. Commutative Archimedean Semigroups of Type 1

Some properties of commutative archimedean semigroups with zero are also studied in the same way as in §3.

Let P be a set, w a mapping $P \to N \setminus \{0\}$. Consider a mapping $P \times P \to N \times N$

$$(p,q) \to (h_p(p,q),\ h_q(p,q))$$

such that $h_p(p,q)$ and $h_q(p,q)$ satisfy (4.1), (4.2), (4.3) and (10) below:

(10) $w(p) - h_p(p,q) = w(q) - h_q(p,q) \geqq 0$ for all $p, q \in P$.

We define L' and ξ as the same as before. Define ξ' as follows:

$$(m,q)\,\xi'\,(n,p) \iff \text{either } m \geqq w(q) \text{ and } n \geq w(p) \text{ or } m = n \text{ and } q = p.$$

Let $\xi_1 = \xi \cup \xi'$. Then ξ_1 is an equivalence relation on L'. Let $L_1 = L'/\xi_1$.

We assume

$$(11)\quad \begin{cases} w(\alpha) = 1 \text{ or } 2 \quad \text{for all} \quad \alpha \in P;\ \text{in particular } w(\iota) = 1. \\ \pi(\iota) = \tilde{\pi}(\iota) = 0. \\ \pi(\alpha) = 1,\ \tilde{\pi}(\alpha) = 1 \text{ or } 2 \quad \text{for all} \quad \alpha \neq \iota. \end{cases}$$

Consider a non-negative valued function $K(\alpha,\beta)$ which satisfies the following conditions:

$$K(\alpha,\beta) = K(\beta,\alpha),\ K(\alpha,\iota) = 1 \quad \text{for all} \quad \alpha,\beta.$$

(12.1) If either $w(\alpha) < w(\alpha') = w(\alpha\cdot\beta)$ or $w(\alpha\cdot\beta) > w(\alpha'\cdot\beta)$, then $K(\alpha,\beta) = 1$.

(12.2) If $w(\alpha) = w(\alpha')$ and if $w(\alpha\cdot\beta) = w(\alpha'\cdot\beta) = 2$ and $h_{\alpha\cdot\beta}(\alpha\cdot\beta,\ \alpha'\cdot\beta) = h_{\alpha'\cdot\beta}(\alpha\cdot\beta,\ \alpha'\cdot\beta) = 2$, then $K(\alpha,\beta) = K(\alpha',\beta) = 1$.

(12.3) If $w(\alpha) = w(\alpha')$, $w(\alpha\cdot\beta) = w(\alpha'\cdot\beta) = 2$ and $h_{\alpha\cdot\beta}(\alpha\cdot\beta,\ \alpha'\cdot\beta) = h_{\alpha'\cdot\beta}(\alpha\cdot\beta,\ \alpha'\cdot\beta) = 1$ or 0, then $K(\alpha,\beta) = K(\alpha',\beta)$.

(12.4) $w((\alpha\cdot\beta)\cdot\gamma) \neq w(\alpha\cdot(\beta\cdot\gamma))$ implies

$$K(\alpha,\beta) + K(\alpha\cdot\beta,\gamma) \geqq w((\alpha\cdot\beta)\cdot\gamma)$$
$$K(\alpha,\beta\gamma) + K(\beta,\gamma) \geqq w(\alpha\cdot(\beta\cdot\gamma)).$$

(12.5) $w((\alpha\cdot\beta)\cdot\gamma) = w(\alpha\cdot(\beta\cdot\gamma))$ implies

$$K(\alpha,\beta) + K(\alpha\cdot\beta,\gamma) \equiv K(\alpha,\beta\gamma) + K(\beta,\gamma) \qquad (\mathrm{mod.}\ N_\ell)$$

where $\ell = w((\alpha\cdot\beta)\cdot\gamma) - h_{(\alpha\cdot\beta)\cdot\gamma}((\alpha\cdot\beta)\cdot\gamma,\ \alpha\cdot(\beta\cdot\gamma))$, and $i \equiv j$ (mod. N_ℓ) means that either $i = j$ if i and j are less than ℓ or $i \geqq \ell$ and $j \geqq \ell$.

(12.6) Either $\alpha\cdot\alpha \neq \alpha$ or $K(\alpha,\alpha) > 0$ for all $\alpha \in P$.

(12.7) For any $\alpha \in P$, $K(\alpha^{(n)}, \alpha) > 0$ for some $n > 0$.

We choose a non-negative integer valued function $K(\alpha,\beta)$, $0 \leq K(\alpha,\beta) \leq 2$ such that (12.1) through (12.7) are satisfied. Define a binary operation on S as in §3. Then S is a commutative archimedean semigroup with zero. Any semigroup of this kind is obtained in this manner. The semigroup is determined by a four-tuple of a tree S of finite length, a semi-highest prime ι, the set P of primes, and a function K satisfying the above condition. Then

$$((S,\iota,P,K))$$

is called a structural system.

We remark that a prime p_o of a partially ordered set L is called semi-highest if and only if

$$\sigma_{p_o}(p_\alpha) \geqq -1 \quad \text{for all primes } p_\alpha \text{ in L.}$$

As in §3, we have to consider the problems with respect to the existence and determination of structural systems $((S,\iota,P,K))$, but it is still open in general.

As mentioned in §2, the study of commutative nil-semigroups, i.e. commutative archimedean semigroups with zero is reduced to that of commutative semigroups satisfying

$$x^2 = 0 \quad \text{for all } x.$$

It is also a useful idea to study the structure of commutative semigroups with zero 0 satisfying $x^2 = 0$ for all x from the different aspect.

5. Other Studies of Commutative Archimedean Semigroups without Idempotent

5.1 Finitely Generated $\mathfrak{N}$-semigroups. An $\mathfrak{N}$-semigroup is a commutative archimedean cancellative semigroup without idempotent (§2). Petrich [8] proved that if an $\mathfrak{N}$-semigroup S is finitely generated, S is power-joined:

For any x, y $\in$ S there are positive integers m, n such that $x^m = y^n$. However, we have in [3]

Theorem 9. An $\mathfrak{N}$-semigroup S is finitely generated if and only if the structure group G_a of S is finite for all a $\in$ S. An $\mathfrak{N}$-semigroup S is power-joined if and only if G_a is periodic for all a $\in$ S.

In Theorem 9, "for all a" can be replaced by "for at least one a".

The two unsolved problems, the determination of I-functions and the isomorphism conditions, for $\mathfrak{N}$-semigroups (§2) can be solved in some sense in the case of finitely generated $\mathfrak{N}$-semigroups.

All the I-functions for a finite abelian group G are determined by the method of [1]; the isomorphism condition can be described in terms of normal standard elements and the normal form of I-functions [5]. An element a is called a normal standard element of S if $|G_a|$ is minimum.

The following theorem is important.

Theorem 10. A semigroup S is a finitely generated $\mathcal{H}$-semigroup if and only if S is a subdirect product of a finite abelian group and a positive integer additive semigroup.

M. Sasaki will report the detailed results on finitely generated $\mathcal{H}$-semigroups [12].

5.2 Finitely Generated Commutative Archimedean Semigroups without Idempotent.

Let S be a commutative archimedean semigroup without idempotent. Apart from the tree order τ_a defined in §3, we define a relation $\leqq$ on S by divisibility:

$$x \leqq y \quad \text{if and only if either} \quad x = yz \quad \text{for some} \quad z \in S \quad \text{or} \quad x = y.$$

If S has either a zero or no idempotent, then $\leqq$ is a partial order. The order $\leqq$ need not be a tree but $\leqq$ is determined by S itself, while the partial order τ_a depends on a but τ_a is of special type, namely, tree. It seems useful to study the relationship between the structure of S and the type of the partial order $\leqq$, but we have not seen such a deep investigation except Tully's work [27].

He determined the structure of commutative archimedean semigroups for which $\leqq$ is a tree with the ascending chain condition.

We would like to mention here that if S is finitely generated or power-joined it has interesting properties. Levin obtained the following results [7]. Theorem 12 is the generalization of Theorem 9.

Theorem 11. Let S be a finitely generated commutative archimedean semigroup without idempotent. Then the partial order $\leqq$ satisfies the ascending chain condition, and the set of all maximal elements of S with respect to $\leqq$ is finite. Also the set of all maximal elements of S with respect to τ_a is finite for each $a \in S$.

Theorem 12. A commutative archimedean semigroup S without idempotent is power-joined if and only if the structure group G_a with respect to an element a is periodic for each $a \in S$, and the congruence class S_ε of S modulo ρ_a which contains the element a is power-joined.

Theorem 13. Let S be a commutative archimedean semigroup without idempotent. Then S is power-joined if and only if every finitely generated subsemigroup is archimedean.

The property "power-joined" (stronger than "archimedeaness") plays an important role in the theory of commutative semigroups. Let S be a commutative semigroup. Consider equivalences σ on S such that each equivalence class of S modulo σ is a subsemigroup. There is a smallest σ_o of such σ on S. Then each equivalence class of S modulo σ_o is power-joined. In other words, every commutative semigroup is the set union of disjoint power-joined subsemigroups.

Recently M. Sasaki [12] has obtained the following results on power-joined $\mathcal{H}$-semigroups. The following theorem is the extension of Theorem 10

Theorem 14. S is a power-joined $\mathcal{H}$-semigroup if and only if it is a subdirect product of a periodic abelian group and an additive positive rational semigroup.

For the readers we define here subdirect product:

If S is a subsemigroup of the direct product of A and B and if the projection of S to A is equal to A and the projection of S to B is equal to B, then S is called a subdirect product of A and B.

We would like to add the results of locally cyclic semigroups [6].

Definition. A semigroup S is called locally cyclic if for every two elements a and b of S there is an element c of S and there are positive integers m and n such that

$$a = c^m, \quad b = c^n.$$

A locally cyclic semigroup is commutative archimedean, power-joined, and if it has no idempotent, it is power-cancellative:

$$a^n = b^n \Rightarrow a = b.$$

A locally cyclic semigroup is characterized by the set union of an ascending chain of cyclic semigroups. More important is that locally cyclic semigroups are related to the additive semigroup of all positive rational numbers. For example a locally cyclic semigroup without idempotent can be embedded into the additive semigroup of all positive rational numbers; more generally

> a power-cancellative, power-joined commutative semigroup can be embedded into the additive semigroup of all positive rational numbers.

We can easily see that a commutative, finitely generated, power-joined power-cancellative semigroup is isomorphic with a positive integer semigroup with addition; we already know in [20] that the positive rational semigroup with addition is also studied from the property "power-divisibility", namely, for any element a and for any positive integer m there is an element b such that

$$a = b^m.$$

It would be interesting that the system of numbers, from positive integers to positive real numbers, may be systematically investigated from the semigroup-theoretical point of view.

6. Generalized Archimedean Semigroups

In this section we will report the results on the study of archimedean semigroups in generalized sense. These results are due to J. Chrislock [3].

Definition. A semigroup S is called $*$-archimedean if, for every a, b $\in$ S there are elements u, v, x, y $\in$ S and positive integers m, n such that

$$a^m = ubv, \quad b^n = xay.$$

S is called right $*$-archimedean if for every a, b $\in$ S there are x, y $\in$ S and positive integers m, n such that

$$a^m = bx, \quad b^n = ay.$$

A (right) $*$-archimedean semigroup is S-indecomposable but the converse is not true.

Theorem 15. A semigroup is a $*$-archimedean semigroup with an idempotent if and only if it is an ideal extension of a simple semigroup with an idempotent by a nil-semigroup.

Theorem 16. A semigroup is a right $*$-archimedean semigroup with an idempotent if and only if it is an ideal extension of a right group by a nil-semigroup.

Definition. If S satisfies

$$xaby = xbay \quad \text{for all} \quad x, a, b, y \in S,$$

then S is called medial.

As archimedeaness is equivalent to S-indecomposability in commutative semigroups, we have

Theorem 17. Let S be a medial semigroup. S is S-indecomposable if and only if it is *-archimedean.

In case of medial semigroups, Theorem 15 is specialized as follows:

Theorem 18. S is a medial *-archimedean semigroup if and only if S is an ideal extension of a rectangular-abelian group by a nil-semigroup where a rectangular-abelian group is the direct product of a rectangular band and an abelian group.

Definition. A semigroup S is called left separative if S satisfies

$$a^2 = ab \quad \text{and} \quad b^2 = ba \quad \text{imply} \quad a = b.$$

S is called right separative if

$$a^2 = ba \quad \text{and} \quad b^2 = ab \quad \text{imply} \quad a = b.$$

S is called separative if it is both left and right separative.

Let S be a medial semigroup and define a relation σ on S by

$x \sigma y$ if and only if there is a positive integer n

such that

$$x^n y = x^{n+1} \quad \text{and} \quad y^n x = y^{n+1}.$$

Then σ is the smallest left separative congruence on S.

We have the results for medial semigroups similar to those for commutative semigroups by Hewitt and Zuckerman (see [1]).

Theorem 19. Let S be a medial semigroup. Then S is left separative if and only if each of its $*$-archimedean components is left cancellative.

Theorem 20. A medial semigroup S can be embedded into a semilattice of groups if and only if S is separative.

Definition. A semigroup which is medial, left separative $*$-archimedean and has no idempotent is called an $\mathfrak{N}^*$-semigroup.

We have a construction-method for $\mathfrak{N}^*$-semigroups similar to that for $\mathfrak{N}$-semigroups.

Theorem 21. Let H be a right-abelian group (i.e. the direct product of a right zero semigroup and an abelian group), I be a non-negative valued function defined on $H \times H$ satisfying the following properties:

(13.1) $I(B,\gamma) + I(\alpha,\beta\gamma) = I(\alpha,\beta) + I(\alpha\beta,\gamma) = I(\beta,\alpha) + I(\beta\alpha,\gamma)$.

(13.2) For any $\alpha \in H$, $I(\alpha^m,\alpha) > 0$ for some $m > 0$.

(13.3) $I(\varepsilon,\varepsilon) = 1$ for some left identity ε.

Define a binary operation on $N \times H = \{(n,\alpha);\ n=0,1,\cdots,\ \alpha \in H\}$ as follows:

$$(n,\alpha)(m,\beta) = (n + m + I(\alpha,\beta),\ \alpha\beta).$$

Then $N \times H$ is an $\mathfrak{N}^*$-semigroup. Every $\mathfrak{N}^*$-semigroup is obtained in this manner.

7. Finite S-indecomposable Semigroups

We can easily treat finite S-indecomposable semigroups in comparison with the general case. The result is published in [23].

Let S be a finite simple or 0-simple semigroup. If S is simple, S is always S-indecomposable; if S is 0-simple, S is S-indecomposable if and only if S contains zero-divisor except zero.

Accordingly we may restrict our study to the case of finite non-simple $\mathfrak{S}$-indecomposable semigroups. All such semigroups are classified into 16 classes as Table 1, 2 shown below. In the tables the example of S with minimal order will be shown in each case.

Explanation of Notations

σ is a smallest C-congruence (commutative congruence) of S,

τ is a smallest $\mathfrak{I}$-congruence (idempotent congruence) of S,

I is a finite simple semigroup,

M is the union of all 0-minimal ideals of S:

$$M = I_1 \cup I_2 \cup \cdots \cup I_k,$$

where $I_1, \cdots, I_k$ are 0-minimal ideals of S and each one is either a null-semigroup or a 0-simple semigroup and

$$I_i I_j = \{0\}, \quad i \neq j.$$

M is called the 0-amalgam of $I_1, \cdots, I_k$.

"c-ind" is "c-indecomposable",

"c-dec" is "c-decomposable",

"nil" is "nil-semigroup",

"u.g." is "unipotent semigroup without zero",

"rect" is "rectangular band",

"null" is "null-semigroup",

A_5 is the alternative group of degree 5,

C_5 is a 0-simple semigroup of order 5 with zero divisor except 0,

N_i is a null-semigroup of order i,

R_2 is the right zero semigroup of order 2,

G_2 is the cyclic group of order 2,

$A_5 \times R_2$ is the direct product of A_5 and R_2,

$N_3 {}^\wedge C_5$ is the 0-amalgam of N_3 and C_5.

Table 1

Non-simple S-indecomposable Semigroups without Zero

I	T	S	S/σ	S/τ	Example I	Example T	Min. Order	Class No.
c-ind group	c-ind	c-ind i-ind			A_5	C_5	64	1.1
	c-dec	c-dec i-ind	nil		A_5	N_2	61	1.2
c-ind i-dec simple	c-ind	c-ind i-dec		rect	$A_5 \times R_2$	C_5	124	1.3
	c-dec	c-dec i-dec	nil	rect	$A_5 \times R_2$	N_2	121	1.4
c-dec group	c-ind	c-dec i-ind	group		G_2	C_5	6	1.5
	c-dec	c-dec i-ind	u.g.		G_2	N_2	3	1.6
c-dec i-dec simple	c-ind	c-dec i-dec	group	rect	$G_2 \times R_2$	C_5	8	1.7
	c-dec	c-dec i-dec	u.g.	rect	$G_2 \times R_2$	N_2	5	1.8

Table 2

Non-simple S-indecomposable Semigroups with Zero

M	T	S	S/σ	Example M	Example T	Min. Order	Class No.
0-amalgam of 0-simple sgs	c-ind	c-ind		C_5	C_5	9	2.1
	c-dec	c-dec	nil	C_5	N_2	6	2.2
Null	c-ind	c-ind		N_3	C_5	7	2.3
		c-dec	nil	N_3	C_5	7	2.4
	c-dec	c-dec	nil	N_2	N_2	2	2.5
0-amalgam of 0-simple and Null sgs	c-ind	c-ind		$N_3 {}^\wedge C_5$	C_5	11	2.6
		c-dec	nil	$N_3 {}^\wedge C_5$	C_5	11	2.7
	c-dec	c-dec	nil	$N_2 {}^\wedge C_5$	N_2	7	2.8

Thus the construction of finite non-simple S-indecomposable semigroups is reduced to the construction of ideal extensions of an S-indecomposable semigroup by an S-indecomposable semigroup with zero. This principle can be discussed from more general point of view [23]. Let $\mathcal{I}$ be a system of identities.

Theorem 22. Let I be a $\mathcal{I}$-indecomposable semigroup and T be a $\mathcal{I}$-indecomposable semigroup with zero. Then every ideal extension S of I by T is $\mathcal{I}$-indecomposable.

Theorem 23. Let I be a $\mathcal{I}$-decomposable semigroup and σ be the smallest $\mathcal{I}$-congruence on I. Let T be a $\mathcal{I}$-indecomposable semigroup. If S is an ideal extension of I by T and if S has a $\mathcal{I}$-congruence ρ such that the restriction $\rho|I$ of ρ to I is equal to σ, then ρ is the smallest $\mathcal{I}$-congruence on S.

Finally we shall review the study of finite nil-semigroups.

The construction of finite nil-semigroups was studied by the author by using the homomorphic image of certain free semigroups [19]. Yamada [28] constructed all commutative nil-semigroups of order n by using the ideal extension of a null semigroup of order 2 by a commutative nil-semigroup of order n-1. On the other hand all commutative nil-semigroups of order n can be constructed as ideal extensions of commutative nil-semigroups of order n-1 by a null semigroup of order 2. Especially we can explicitly determine all the non-isomorphic commutative extensions of a cyclic nil-semigroup by a null semigroup of order 2, and all the non-isomorphic commutative extensions of a finite null semigroup by a null semigroup of order 2. These results were obtained by Yamada and the author [29].

8. S-indecomposable semigroups

It would be difficult to characterize or construct S-indecomposable semigroups as a class of semigroups because the class is very large.

The following statement is clear from the definition.

A semigroup S is S-indecomposable if and only if for every two contents A, B in S, there is a sequence of contents in S

$$A = C_1, C_2, \cdots, C_n = B$$

such that

$$C_i \cap C_{i+1} \neq \emptyset \ (i=1,\cdots,n-1).$$

The concept of "content" may be useful in the study of S-indecomposable semigroups.

The greatest C-homomorphic image of an S-indecomposable semigroup is archimedean and the greatest $\mathcal{S}$-homomorphic image of an S-indecomposable semigroup is a rectangular band. A subdirect product of a rectangular band and a commutative archimedean semigroup is S-indecomposable, but all S-indecomposable semigroups are not subdirect products of a rectangular band and a commutative archimedean semigroup, though they are homomorphic onto a subdirect product.

The deep study of S-indecomposable semigroups is left for the future.

9. Subdirect Product

So far the concept of subdirect product has been used frequently to describe the structure and to construct certain class of semigroups.

Although the direct product of A and B is uniquely determined by A and B, subdirect products of A and B are not necessarily unique. How can we concretely construct all subdirect products of given A and B? Of course the following statement is clear.

Let ρ and σ be congruences of a semigroup S. S is isomorphic with a subdirect product of S/ρ and S/σ if and only if $\rho \cap \sigma = \iota$, (ι equality relation).

We have not studied the construction of subdirect products in general case except a special case [4].

Theorem 24. A subdirect product of an S-indecomposable semigroup and a rectangular band is S-indecomposable.

Theorem 25. Let S be a semigroup and $B = L \times R$ be a rectangular band. If $\mathfrak{L}$ is the set of all left ideals of S and $\mathfrak{R}$ is the set of all right ideals of S, then two mappings $\varphi: L \to \mathfrak{R}$ and $\psi: R \to \mathfrak{L}$ satisfying

(14) $S = \bigcup\{\varphi(\lambda);\ \lambda \in L\} = \bigcup\{\psi(\rho);\ \rho \in R\}$

determine a subdirect product $D \subseteq S \times B$ by

$$D = \bigcup\{D(\lambda,\rho);\ (\lambda,\rho) \in B\}$$

where

$$D(\lambda,\rho) = \{(x;\lambda,\rho);\ x \in \varphi(\lambda) \cap \psi(\rho)\}.$$

Moreover the correspondence $(\varphi,\psi) \to D$ is one-to-one onto the set of all subdirect products of S and B.

We remark that the direct product of a finite number of S-indecomposable semigroups is S-indecomposable, but the direct product of an infinite number of S-indecomposable semigroups need not be so. Also a subdirect product of two S-indecomposable semigroups need not be so.

10. General Products

We frequently meet with the problem: Find semigroups S which are homomorphic onto a given semigroup T. The problem in this form seems too vague to treat. So let us restrict the problem as follows:

> Given T, find S such that S is homomorphic onto T under a map f and such that the cardinal number of the inverse image set of each element of T is constant, i.e. given $\mathfrak{m}$
>
> $$|tf^{-1}| = \mathfrak{m} \quad \text{for all} \quad t \in T.$$

We will introduce a concept "general product" of a set A by a semigroup T by using the system of groupoids.

10.1 The System of Binary Operations. The author studied the system of semigroup operations on a set in 1952 [13]. Here consider the set of all binary operations (i.e. groupoids) defined on a set.

Let E be a set and $\mathscr{B}_E$ be the set of all binary operations defined on E. Let $x, y \in E$, $\theta \in \mathscr{B}_E$ and let $x \theta y$ denote the product of x and y by θ. A groupoid with θ defined on E is denoted by $E(\theta)$. First, define the equality of elements of $\mathscr{B}_E$

$$\theta = \eta \quad \text{if and only if} \quad x\theta y = x\theta y \quad \text{for all} \quad x, y \in E.$$

Let $a \in E$ be fixed. Two binary operations a^* and $^*a$ are defined in $\mathscr{B}_E$ as follows:

$$\theta, \eta \in \mathscr{B}_E, \quad x, y \in E$$

(15) $x(\theta\, a^* \,\eta)y = (x\, \theta\, a)\eta\, y$,

(16) $x(\theta\, {}^*a\, \eta)y = x\, \theta\, (a\, \eta\, y)$.

Then $\mathscr{S}_E$ is a semigroup with respect to a^* and ${}^*a$ for all $a \in E$;

(17.1) $\mathscr{S}_E(a^*) \cong \mathscr{S}_E(b^*)$ for all $a, b \in E$,

(17.2) $\mathscr{S}_E({}^*a) \cong \mathscr{S}_E({}^*b)$ for all $a, b \in E$,

(17.3) $\mathscr{S}_E(a^*)$ is anti-isomorphic with $\mathscr{S}_E({}^*a)$,

(17.4) $E(\theta)$ is a semigroup if and only if $\theta\, a^*\, \theta = \theta\, {}^*a\, \theta$ for all $a \in E$.

10.2 Basic Properties of General Product. Let S be a set and T be a semigroup. Let

$$S \times T = \{(x,\alpha);\ x \in S,\ \alpha \in T\}$$

Consider a mapping Θ of $T \times T$ into $\mathscr{S}_S$

$$(\alpha,\beta)\Theta = \theta_{\alpha,\beta} \qquad (\alpha,\beta) \in T \times T$$

satisfying

(18) $\theta_{\alpha,\beta}\, a^*\, \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\beta\gamma}\, {}^*a\, \theta_{\beta,\gamma}$ for all $a \in S$, all $\alpha, \beta, \gamma \in T$.

Given S, T, Θ, a binary operation is defined on $S \times T$ as follows:

(19) $(x,\alpha)(y,\beta) = (x\, \theta_{\alpha,\beta} y,\ \alpha\beta)$.

Then $S \times T$ is a semigroup with respect to (19).

Definition. The semigroup, $S \times T$ with (19), is called a general product of a set S by a semigroup T with respect to Θ, and is denoted by

$$S \times_\Theta T.$$

If it is not necessary to specify Θ, it is denoted by

$S \times T$.

Definition. Let g be a homomorphism of a semigroup D onto a semigroup T: $D = \bigcup_{\alpha \in T} D_\alpha$, $D_\alpha g = \alpha$. If $|D_\alpha| = |D_\beta|$ for all $\alpha, \beta \in T$, then g is called a homogeneous homomorphism (h-homomorphism) of D or D is said to be h-homomorphic onto T. If $|D_\alpha| > 1$ and $|T| > 1$, then g is called a proper h-homomorphism.

Definition. If a semigroup D is isomorphic onto some $S \times_\Theta T$, then D is called general product decomposable (gp-decomposable). If $|S| > 1$ and $|T| > 1$, then D is called properly gp-decomposable.

Theorem 26. A semigroup D is (properly) gp-decomposable if and only if D has a (proper) h-homomorphism.

Theorem 27. A semigroup D is gp-decomposable if and only if there is a congruence ρ on D and an equivalence σ on D such that

(20) $\rho \circ \sigma = \omega$, $\omega = D \times D$

(21) $\rho \cap \sigma = \iota$, $\iota = \{(x,x);\ x \in D\}$

in which (20) can be replaced by

(20') $\sigma \circ \rho = \omega$.

Then $D \cong (D/\sigma) \times (D/\rho)$.

We know many examples of general product: direct product, semi-direct product, group extension, Rees' regular representation of completely simple semigroups, the representation of $\mathcal{N}$-semigroups and so on.

10.3 Left General Product.

Definition. A general product $S \times_\Theta T$ is called a left general product of S by T if

$$\theta_{\alpha,\beta} = \theta_\alpha. \quad \text{depending only on } \alpha.$$

A right general product of S by T if

$$\theta_{\alpha,\beta} = \theta_{\cdot\beta} \text{ depending only on } \beta.$$

Theorem 28. A semigroup D is isomorphic onto a left general product of a set S by T if and only if there is a congruence ρ on D and a left congruence σ on D such that

$$D/\rho \cong T, \quad |D/\sigma| = |S|,$$

and

$$\rho \cdot \sigma = \omega,$$
$$\rho \cap \sigma = \iota.$$

It can be proved that $\mathscr{F}_E(a^*)$ can be described as a left general product of S by T where T is the full transformation semigroup over E.

10.4 Sub-general Product.

Definition. If U is a subsemigroup of $S \times_\theta T$ and if the projection of U to T is equal to T, then U is called a subgeneral product of $S \times_\theta T$.

This concept is a generalization of sub-direct product.

Theorem 29. If a semigroup D is homomorphic onto a semigroup T under a mapping g, then D is isomorphic into $S \times_\theta T$.

We can restrict the isomorphism to a certain strict sense. It should be noted that there is S_o among the above S such that $|S_o|$ is either the minimum of $|S|$ or the minimum plus one. We do not know yet that S_o can be taken such that $|S_o|$ is always minimal.

The above remark on $|S_o|$ is important. If we did not assume this, Theorem 29 would be trivial, because if D is homomorphic onto T, then D is isomorphic with a subdirect product of D and T.

10.5 Examples and Remark.

Let T be a right zero semigroup. To find all left general products of given S and T, we may find $\{\theta_{\alpha\cdot}; \alpha \in T\}$ such that the following equations are satisfied:

(21) all θ_α are semigroup operations

(22) $\theta_{\alpha\cdot}\ a{*}\ \theta_{\beta\cdot} = \theta_{\alpha\cdot}\ {*}a\ \theta_{\beta\cdot}$ for all $a \in S$, all $\alpha, \beta \in T$.

As simplest examples, the author's students have computed all left general products of S by T where $|S| \leqq 3$, $|T| = 2$, and they are computing all general products of S by T, $|S| \leqq 3$, $|T| = 2$.

Let T be a right (left) zero semigroup, and suppose that a system of semigroups, $\{D_\alpha; \alpha \in T\}$, is given. If a semigroup D is a set union of D_α and the operation within each D_α is preserved, D is called a r-(ℓ-)composition of $\{D_\alpha; \alpha \in T\}$. R. Yoshida began with the study of ℓ-composition [30]. The following two questions are open:

Under what condition on $\{D_\alpha; \alpha \in T\}$ does there exist D?

How can we find all such D for a given $\{D_\alpha; \alpha \in T\}$?

Instead of a right (left) zero semigroup, if we regard T as a semilattice, D is called a semilattice-composition (S-composition) of $\{D_\alpha; \alpha \in T\}$. The same questions are raised for S-composition.

If we speak of the problems by using Theorem 29, an S-(ℓ-, r-) composition D of $\{D_\alpha; \alpha \in T\}$ exists if and only if a general product $S \times_\Theta T$ exists for some S and some Θ such that $|S| \geq |D_\alpha|$ and D can be embedded into $S \times_\Theta T$. We do not know, however, if this idea will be practically useful for the theory of compositions, but we can say at least that it would be theoretically interesting. The development is expected. The results in this section §10 will be more precisely reported in [25], [26].

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Unsolved Problems on Semigroups

By T. Tamura

1. A commutative archimedean semigroup without idempotent is determined by a structural system $((G, I; \{S_\lambda\}, \{z_\lambda\}; P))$. Solve the existence problem and the construction problem of structural systems. (For example under what condition on $\{S_\lambda\}$, $\{z_\lambda\}$, G, I does there exist a commutative groupoid P such that $((G, I; \{S_\lambda\}, \{z_\lambda\}; P))$ is a structural system?) (See T. Tamura, Recent study of semigroups, ibid.) Also solve the isomorphism condition.
2. The same problem for a commutative nil-semigroup. In this case $((S_\varepsilon, z_\varepsilon, P))$ is a structural system.
3. Study the structure of commutative semigroups S with zero 0 such that $x^2 = 0$ for all $x \in S$. Especially try it without using the notion "tree". Can we determine the structure of S from the standpoint of the partial order defined by divisibility? The same question for 1.
4. Let S be a semigroup and suppose that $f: S \to L$ is a homomorphism of S onto a semilattice L. Let φ be a right translation of S. Then it is known that $xf = yf$, $x, y \in S$, implies $(x\varphi)f = (y\varphi)f$. For φ, a translation $\bar{\varphi}$ of L is well defined by

$$(xf)\bar{\varphi} = (x\varphi)f.$$

Then the mapping $\varphi \to \bar{\varphi}$ is a homomorphism of the right translation semigroup $R(S)$ into the translation semigroup $T(L)$. ($T(L)$ is a semilattice.) A question: Is $\varphi \to \bar{\varphi}$ a homomorphism of $R(S)$ onto $T(L)$?

5. Let $\mathfrak{S}_S$ denote the symmetric group over a set S, and G be a groupoid defined on S. Let $\mathfrak{P}$ be a permutation group over S, that is, a subgroup of $\mathfrak{S}_S$, and $G(\mathfrak{P})$ denote a groupoid whose automorphism group is $\mathfrak{P}$. What is a necessary and sufficient condition on $\mathfrak{P}$ for $G(\mathfrak{P})$ to exist? Consider the same problem when $G(\mathfrak{P})$ is restricted to semigroups. We know that $G(\mathfrak{P})$ exists if one of the following conditions is satisfied: (1) $\mathfrak{P} = \mathfrak{S}_S$, (2) $\mathfrak{P} = \{\varepsilon\}$ (the identity mapping alone), (3) $\mathfrak{P}$ is a cyclic subgroup, (4) every permutation group over S, $|S| \leqq 4$. On the other hand $G(\mathfrak{P})$ does not exist if $|S| \geqq 5$, $\mathfrak{P} \neq \mathfrak{S}_S$ and if $\mathfrak{P}$ is triply transitive. (See T. Tamura, Groupoids and their automorphism groups (to be published).)

6. Similarly to 5, consider the problem: What is a relationship between a groupoid G and its endomorphism semigroup?

7. "Attainability" is defined in Jour. of Algebra, 3(1966), 261-276.

(7.1) Determine all non-trivial attainable system of identities (admitting constant letters) on all semigroups.

(7.2) Determine all non-trivial attainable system of implications on all semigroups.

(7.3) Is there non-trivial system of identities on all groupoids?

(It is known that there is no such system of identities on all rings.) $\{x^2 = xy = y^2 \Rightarrow x = y\}$ (separative) is attainable on all commutative semigroups; weak reductivity is attainable on all semigroups.

(7.4) Is $\{x^2 = xy = y^2 \Rightarrow x = y\}$ attainable on all semigroups?

(7.5) Is $\{\{x^2 = x, y^2 = y\} \Rightarrow x = y\}$ attainable on all semigroups?

8. All the identities containing at most three letters in bands were determined by M. Yamada and N. Kumura. Determine all the identities containing at most four letters in bands. What about the general case?

9. A semigroup S is called totally orderable if a total order $\leq$ can be defined in S such that

$$a \leq b \Rightarrow ax \leq bx,\ xa \leq xb.$$

Find a necessary and sufficient condition on S for S to be totally orderable.

10. Let S be a semigroup, and $\mathfrak{P}(S)$ be the set of all non-empty subsets of S. A binary operation is defined in $\mathfrak{P}(S)$ as follows: For X, $Y \in \mathfrak{P}(S)$

$$X \cdot Y = \{xy;\ x \in X,\ y \in Y\}.$$

Then $\mathfrak{P}(S)$ is a semigroup. It is called the power-semigroup of S. A question is this: Is it true that

$$\text{if } \mathfrak{P}(S_1) \cong \mathfrak{P}(S_2),\ S_1 \cong S_2\ ?$$